On the superintegrability of Calogero-Moser-Sutherland model

1998 J. Phys. A: Math. Gen. 31 4465

(http://iopscience.iop.org/0305-4470/31/19/012)

J. Phys. A: Math. Gen. **31** (1998) 4465–4472. Printed in the UK PII: S0305-4470(98)86902-3

On the superintegrability of Calogero–Moser–Sutherland model

Cezary Gonera

Department of Field Theory, University of Łódź, Pomorska 149/153, 90–236 Łódź, Poland

Received 18 August 1997, in final form 12 January 1998

Abstract. The superintegrability of Calogero–Moser–Sutherland hyperbolic model and nonsuperintegrability of trigonometric one are shown.

1. Introduction

Integrability and interesting links to various branches of physics make the Calogero–Moser–Sutherland (CMS) model [1–4], although formulated almost 30 years ago, still attract much attention.

It has been observed to be relevant in context of two-dimensional (2D) gravity [5], 2D QCD [6], fractional statistics [7], quantum Hall effect [8], spin chains [9] and others. Classical and quantum integrability, very important and specific feature of the CMS model, has been analysed in the framework of different techniques: inverse scattering method [10], R-matrix formalism [11], collective field method [12], W-algebra techniques [13].

Suprisingly enough, it has appeared that the rational CMS model (with and without harmonic potential) provides an example of a rare and more peculiar type of system than integrable ones. It has been shown to be superintegrable both on the classical and quantum level [14–16]. In the case of N degrees of freedom this means that in addition to N global, functionally independent integrals of motion in involution there exist further $N-1$ global, functionally independent integrals of motion not depending explicitly on time. Classically, an intersection of the level surfaces of all $2N-1$ integrals determines uniquely the trajectories of the system in the phase space. In the case of the compact energy surface these trajectories are closed and motion is strictly periodic. The harmonic oscillator and Kepler problem are well known examples of such a system. At the quantum level the superintegrability is related to degeneration of energy.

In this paper we show, referring to the projection method, the superintegrability of the hyperbolic CMS model governed by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{g^2}{2}\sum_{i\neq j}\frac{a^2}{\sinh^2 a(q_i - q_j)} \tag{1}$$

and nonsuperintegrability of the trigonometric CMS model, the Hamiltonian of which can be obtained from the previous one by replacing $a \to \mathrm{i}a$. These results obviously imply the nonsuperintegrability of general elliptic CMS model where potential is given by the Weierstrass function.

0305-4470/98/194465+08$19.50

One may wonder why the two models, related to each other by simple analytic continuation, exhibit such different behaviour—one of them is superintegrable while the other is not. However, this can be understood as follows. The integrability means that there exist n globally defined integrals of motion $F_i, i = 1, \ldots, n$ which are in involution. If the resulting hypersurface $F_i = \text{constant}$, $i = 1, \ldots, n$ is compact (and connected, but this assumption is not critical), the Arnold–Liouville theorem [17] implies that it is a (n-dimensional) torus. If one now takes specific combinations of the integrals F_i (i.e. action variables) as new momenta then the canonically conjugated position variables (angle variables) are not globally well-defined functions on phase space (they are just angles parametrizing the Arnold–Liouville tori); only the trigonometric functions of them are well defined. The angle variables have linear time dependence

$$\varphi_I(t) = \omega_i(J)t + \varphi_i(0). \tag{2}$$

One can therefore construct out of them n independent but time-dependent integrals of the form:

$$\varphi_i - \omega_i(J)t = \varphi_i(0). \tag{3}$$

Now, in order to obtain $N - 1$ time-independent integrals one has to eliminate explicit time dependence. It is easy to see [18] that the resulting integrals are not globally defined (and no functions of them are) unless all rations $\omega_i(J)/\omega_k(J)$ are rational numbers; in the latter case one obtains $N - 1$ additional integrals by taking appropriate trigonometric functions (this is nicely discussed in [18]). Let us now assume that the hypersurface $F_i = \text{constant}$ is noncompact and topologically trivial. Then, in general, there is no obstruction for the canonically conjugated 'angle' variables to be globally well defined and the procedure leading to time-independent global integrals of motion is straightforward; in particular no condition on ratios $\omega_i(J)/\omega_k(J)$ is necessary. Let us stress that if two systems (compact and noncompact ones) are related by simple transformation of variables they both possess (or do not possess) the same number of local integrals of motion; however, in the compact case some of them may fail to be globally defined despite the fact that their noncompact counterparts are globally defined.

Let us give a simple example illustrating the above remarks. Consider two uncoupled harmonic oscillators:

$$H = \left(\frac{p_1^2}{2m} + \frac{m\omega_1^2}{2}x_1^2\right) + \left(\frac{p_2^2}{2m} + \frac{m\omega_2^2}{2}x_2^2\right). \tag{4}$$

The general solution to the equations of motion reads

$$\begin{aligned} x_i &= A_i \sin(\omega_i t + \varphi_i) \\ p_i &= m\omega_i A_i \cos(\omega_i t + \varphi_i) \qquad i = 1, 2 \end{aligned} \tag{5}$$

this system is obviously integrable, the relevant integrals of motion being the energies H_i of both oscillators. These integrals are related to the A_i constants

$$A_i = \sqrt{\frac{2}{m\omega_i^2}H_i}. \tag{6}$$

The remaining integrals read

$$\varphi_i = \arcsin\frac{x_i}{\sqrt{\frac{2}{m\omega_i^2}H_i}} - \omega_i t \tag{7}$$

so there exists one additional time–independent local integral

$$\omega_1\varphi_2 - \omega_2\varphi_1 \qquad \text{or} \qquad \varphi_2 - \frac{\omega_2}{\omega_1}\varphi_1. \tag{8}$$

However, it is not defined globally and, due to the fact that the replacement $\varphi_i \to \varphi_i + 2\Pi n_i$ results in the change $\varphi_2 - \frac{\omega_2}{\omega_1}\varphi_1 \to \varphi_2 - \frac{\omega_2}{\omega_1}\varphi_1 + 2\Pi(n_2 - \frac{\omega_2}{\omega_1}n_1)$ no function of it is globally defined unless ω_2/ω_1 is rational. In the latter case we can take, for example,

$$\sin(k\varphi_2 - l\varphi_1) \qquad \frac{\omega_2}{\omega_1} = \frac{k}{l} \tag{9}$$

as an additional integral. Let us now make a simple replacement $x_i \to \mathrm{i}x_i$, $t \to \mathrm{i}t$. Then

$$H \to \left(\frac{p_1^2}{2m} - \frac{m\omega_1^2}{2}x_1^2\right) + \left(\frac{p_2^2}{2m} - \frac{m\omega_2^2}{2}x_2^2\right) \tag{10}$$

and the solution to the modified equations of motion read

$$\begin{aligned} x_i &= A_i \mathrm{e}^{\omega_i t} + B_i \mathrm{e}^{-\omega_i t} \\ p_i &= m\omega_i (A_i \mathrm{e}^{\omega_i t} - B_i \mathrm{e}^{-\omega_i t}). \end{aligned} \tag{11}$$

As in the ‘confining’ case both energies are globally defined over the phase space; however, in this case one can construct additional global integral of motion. It reads

$$\omega_2 \ln\left(x_1 + \frac{p_i}{m\omega_1}\right)^2 - \omega_1 \ln\left(x_2 + \frac{p_2}{m\omega_2}\right)^2. \tag{12}$$

Let us note that both integrals (9), (12) are defined locally except the hypersurface $H_1 \cdot H_2 = 0$. However, the integral (12) is defined globally irrespectively of the value that the ratio ω_1/ω_2 takes. Turning the argument around lets us return to the integral (12), to the confining case. It amounts in the replacement $x_i \to -\mathrm{i}x_i$ in equation (12). Both logarithms are then defined up to a multiple of 2Π and again the value of ω_1/ω_2 becomes crucial.

Returning to the general case let us also note that, in the compact case the superintegrability is equivalent to the periodicity of all trajectories [18, 19]. Indeed, any globally defined dynamical variable can be expanded in Fourier series in angle variables with coefficients depending on action variables. The periodicity condition is then equivalent to the statement that all ratios ω_i/ω_k are rational so the additional integrals of motion can be constructed according to the method given above. Conversely, the intersection of $2N - 1$ surfaces (in general position) corresponding to the time-independent integrals is one-dimensional (1D) closed submanifold of Arnold–Liouville torus, i.e. it is a compact 1-manifold. Therefore each of its connected component is diffeomorphic to a circle.

To conclude the introduction we present a short review of the reduction method as applied to hyperbolic/trigonometric CMS model [3]. First we separate the centre-of-mass motion. For the remaining degrees of freedom the solution to the Hamiltonian equations are obtained by projecting from the simple dynamics defined on the homogeneous space $SL(\mathbb{N}|\mathbb{C})/SU(N)$ of Hermitian positive definite $\mathbb{N} \times \mathbb{N}$ matrices with unit determinant. The dynamical flow on this space is given by the equations for geodesics (with respect to $SL(\mathbb{N}, \mathbb{C})$-invariant metric on $SL(\mathbb{N}, \mathbb{C})/SU(\mathbb{N})$; $\mathrm{d}s^2 = \mathrm{Tr}(x^{-1}\mathrm{d}x\, x^{-1}\mathrm{d}x)$),

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^{-1}\dot{x} + \dot{x}x^{-1}) = 0. \tag{13}$$

(The dot means differentation with respect to time.) In order to obtain the relevant solution to the CMS model the special form of geodesics is considered

$$x(t) = b\mathrm{e}^{2Vt}b^{+}b \in SL(\mathbb{N}, \mathbb{C}) \qquad V^{+} = V \qquad \mathrm{Tr}\, V = 0. \tag{14}$$

The Hermitian matrices $x(t)$ can be diagonalized by time-dependent unitary transformation $u(t)$

$$\begin{aligned} & x(t) = u(t)\mathrm{e}^{2aQ(t)}u^{-1}(t) \\ & u(t) \in SU(N)\, Q(t) = \mathrm{diag}(q_1(t), \ldots, q_N(t)). \end{aligned} \tag{15}$$

It can then be shown that $q_i(t), i = 1, \ldots, N$ provide a solution to hyperbolic CMS model (the trigonometric case can be obtained by replacing $a \to \mathrm{i}a$). The proof relies on the following equations which can be easily derived

$$\tfrac{1}{2}(\dot{x}x^{-1} + x^{-1}\dot{x}) = 2au(t)L(t)u^{-1}(t) = bVb^{-1} + (b^{+})^{-1}Vb^{+} \tag{16a}$$

$$\tfrac{1}{2}(\dot{x}x^{-1} - x^{-1}\dot{x}) = iu(t)K(t)u^{-1}(t) = bVb^{-1} - (b^{+})^{-1}Vb^{+} \tag{16b}$$

where

$$L(t) = P(t) + \frac{\mathrm{i}}{4a}\left(\mathrm{e}^{-2aQ(t)}M(t)\mathrm{e}^{2aQ(t)} - \mathrm{e}^{2aQ(t)}M(t)\mathrm{e}^{-2aQ(t)}\right) \tag{17a}$$

$$K(t) = M(t) - \tfrac{1}{2}\left(\mathrm{e}^{2aQ(t)}M(t)\mathrm{e}^{-2aQ(t)} + \mathrm{e}^{-2aQ(t)}M(t)\mathrm{e}^{2aQ(t)}\right) \tag{17b}$$

$$M(t) = -iu^{-1}(t)\dot{u}(t) \tag{17c}$$

$$P(t) = \dot{Q}(t). \tag{17d}$$

With the help of the above relations one obtains the following Lax-type equations

$$\mathrm{i}\dot{L} = [M, L] \tag{18a}$$

$$\mathrm{i}\dot{K} = [M, K]. \tag{18b}$$

In fact the second Lax equation (18b) is trivial because, as it follows from equation (16b) K is just the value of moment map and, therefore, is a constant. Now, specifying, L, M matrices to be the Lax-pair for hyperbolic CMS model

$$L_{jk} = p_j\delta_{jk} + \mathrm{i}(1 - \delta_{jk})ag\coth(q_j - q_k) \tag{19a}$$

$$M_{jk} = ga^2\left(\delta_{jk}\sum_{i\neq j}\frac{1}{\sinh^2 a(q_j - q_i)} - (1 - \delta_{jk})\frac{1}{\sinh^2 a(q_j - q_k)}\right) \tag{19b}$$

one finds that

$$K_{jk} = 2ga^2(1 - \delta_{jk}) \tag{20}$$

is indeed time independent and relations (17) and (18) are fulfilled. Finally, note that $b = \mathrm{e}^{aQ(0)}$ are Hermitian.

2. Superintegrability of the hyperbolic CMS model

In order to show the superintegrability of the hyperbolic CMS model we refer to the reduction method as described in the introduction. We will construct additional integrals of motion from the quantities

$$N_n(t) \equiv \mathrm{Tr}((2aL(t) + \mathrm{i}K(t))^n\mathrm{e}^{2aQ(t)}) \qquad n = 0, 1, 2, \ldots. \tag{21}$$

By equations (14), (15) and the relation

$$bVb^{-1} = u(t)\left(aL(t) + \frac{\mathrm{i}}{2}K(t)\right)u^{-1}(t) \tag{22}$$

which follows from equations (16a) and (16b) we can also write

$$N_n(t) = \mathrm{Tr}((2bVb^{-1})^n b\mathrm{e}^{2Vt}b) \qquad n = 0, 1, 2, \ldots. \tag{23}$$

This implies

$$\dot{N}_n(t) = \mathrm{Tr}((2bVb^{-1})^n(2bVb^{-1})b\mathrm{e}^{2Vt}b) = N_{n+1}(t). \tag{24}$$

Equations (24) provide an infinite chain of simple differential equations for $N_n(t)$'s. However, they cannot all be independent because we are dealing with finite-dimensional matrices. Indeed, multiplying the characteristic equation for matrix $2bVb^{-1}$ by $b\mathrm{e}^{2Vt}b$ and taking the trace we obtain

$$\sum_{n=0}^{N} c_{N-n}N_n(t) = 0 \tag{25}$$

where as it is shown in the appendix

$$c_n = (-1)^{N-n}(2a)^n J_n \tag{26}$$

and J_n's are well known [3, 20, 21] integrals of motion enjoying no ordering ambiguities in quantum theory and constructed out of another form of Lax matrix for CMS hyperbolic model

$$\tilde{L}_{ij} = p_i\delta_{ij} + \mathrm{i}ag(1-\delta_{ij})\frac{1}{\sinh a(q_i - q_j)}. \tag{27}$$

Hence, our infinite chain of equations (24) reduces to the following set of N linear differential equations with constant coefficients

$$\begin{pmatrix} \dot{N}_0 \\ \dot{N}_1 \\ \vdots \\ \dot{N}_{N-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{N-1} \end{pmatrix} \begin{pmatrix} N_0 \\ N_1 \\ \vdots \\ \vdots \\ N_{N-1} \end{pmatrix} \tag{28}$$

$$a_n = (-1)^{N+1}c_{N-n}. \tag{29}$$

The general solution to these equations reads

$$N_k(t) = \sum_{i=0}^{N-1} d_i(2a\lambda_i)^k \mathrm{e}^{2a\lambda_i t} \tag{30}$$

where λ_i are the eigenvalues of the Lax matrix $\tilde{L}$ and d_i's are constants related to initial conditions $N_k(0)$ imposed on functions $N_k(t)$.

In order to show that $N_k(t)$ are globally defined time-dependent functions on the phase space we have to show that λ_i's are globally defined over the (allowed part of) phase space. To this end we note that λ_i's are the roots of characteristic polynomial of Lax matrix; the coefficients of the polynomial are regular function of canonical variables. Therefore, by implicit function theorem, it is sufficient to show that the derivative of the polynomial with respect to λ does not vanish for any $\lambda = \lambda_i$, i.e. our polynomial does not possess multiple roots. This can be shown by considering the $t \to \infty$ limit; then $p_i \to p_i^+ = \lambda_i$ (up to renumbering). The eigenvalues λ_i coincide nowhere due to the fact that for $i \neq k$, $p_i^+ \neq p_k^+$. Hence λ_i's are globally defined regular functions on (allowed region of) phase space. The form of functions $N_k(t)$ as defined by equation (21)

$$N_k(t) = \sum_{i=1}^{N}(2ap_i)^k \mathrm{e}^{2aq_i} + \text{terms of lower order in } p_i\text{'s} \tag{31}$$

implies that they are functionally independent.

Let us note in passing that up to now everything has been done in the centre-of-mass coordinate system (where theWronskian of equations (28) is a constant of motion). However, we can use the Galilean invariance of the model to argue that our result is valid in the general case.

Now, solving equations (29) with respect to d_i's we obtain integrals of motion globally defined, functionally independent but in general depending explicitly on time.

To find time-independent integrals of motion let us call $\mathcal{A}$ the matrix appearing on the RHS of equation (28) and construct operators projecting on eigenspaces of this matrix

$$\pi_k \equiv \frac{\prod_{i\neq k}(\mathcal{A} - 2a\lambda_i)}{(2a)^{N-1}\prod_{i\neq k}(\lambda_k - \lambda_i)}. \tag{32}$$

These are regular, globally defined functions on the phase space because $\mathcal{A}$ and λ_i's are such functions. It follows from equation (31) that

$$(\pi_k N(t))_i = d_k(2a\lambda_k)^i \mathrm{e}^{2a\lambda_k t}. \tag{33}$$

So, as globally defined, functionally independent integrals of motion not depending explicitly on time one can take functions:

$$\begin{aligned} S_n &= \lambda_0 \ln((\pi_n N(t))_0)^2 - \lambda_n \ln((\pi_0 N(t))_0)^2 \\ &= \lambda_0 \ln d_n^2 - \lambda_n lnd_0^2 \qquad n = 1, \ldots, N-1. \end{aligned} \tag{34}$$

They provide additional (i.e. apart from $J_1, \ldots, J_N$) globally defined functionally independent (because d_k's coefficients are arbitrary which in turn follows from functional indepence of $N_k(0)k = 0, \ldots, N-1$). This proves the superintegrability of the CMS hyperbolic model.

3. The trigonometric case

As it has been explained in some detail in the introduction, the superintegrability in the compact case is equivalent to the periodicity of all trajectories. The following precise argument besed on this remark shows the nonsuperintegrability in trigonometric case. One can repeat all steps of section 2, leading to formula (30), with the replacement $a \to \mathrm{i}a$. Formula (30) then becomes the Fourier series for N_k(t) (note that λ_i's, being the eigenvalues of Hermitean Lax matrix, are real also in the trigonometric case). In order to show that N_k(t) are, in general, not periodic it is sufficient to prove that λ_i's are functionally independent; this implies that the ratios λ_i/λ_k are not generally rational. However, it is well known (in fact, it is just the statment that the model is integrable) that the traces $\mathrm{Tr}\, L^k$, $k = 1, \ldots, N$ are functionally independent which immediately implies functional independence of λ_i's. Therefore, N_k(t) are, in general, not periodic and the system under consideration is not superintegrable. As in the simple example presented in section 1, integrals (34), with a replaced by $\mathrm{i}a$, cease to be globally defined.

One should keep in mind the fact that the following argument shows immediately that the number of globally defined integrals of motion cannot exceed $2N-2$. Due to the fact that the total momentum is conserved the centre-of-mass moves freely on the circle (with arbitrary velocity). Therefore the corresponding frequency can take arbitrary values.

Another obvious argument in favour of nonsuperintegrability can be given by taking the $g \to 0$ limit. We are then dealing with the system of N free particles moving on a circle and the generic trajectories are evidently not periodic.

To conclude we expect that similar methods should allow us to prove a superintegrability of generalized CMS hyperbolic models based on other root systems. This will be considered elsewhere.

Acknowledgments

I would like to thank P Kosiński for enlightening discussions and suggestions. Interesting remarks of S Giller, P Maślanka, B Nowak and M Majewski are also acknowledged. Unknown referees are acknowledged for essential remarks which served for substantial improvement of the paper. CG was supported by KBN grant 2 P03B 076 10.

Appendix

In order to find the explicit form of coefficients of characteristic equations (25) for matrix $2bVb^{-1}$:

$$\sum_{n=0}^{N} c_{N-n}(2bVb^{-1})^n = 0 \qquad c_0 = (-1)^N. \tag{35}$$

Let us first note that due to equation (22) we have

$$\sum_{n=0}^{N} c_{N-n}\lambda^n = \det(2bVb^{-1} - \lambda I) = \det(2V - I\lambda) = \det(2aL + \mathrm{i}K - \lambda I). \tag{36}$$

Now, using equation (16*a*) one finds

$$bVb^{-1} + b^{-1}Vb = 2aU(t)L(t)U^{-1}(t) = 2aU(0)L(0)U(0) = 2aL(0). \tag{37}$$

Solving this equation with respect to V one obtains

$$V = a\tilde{L}(0) \tag{38}$$

where $\tilde{L}$ is another form of Lax matrix for hyperbolic CMS model given by equation (27).

Hence

$$\det(2aL + \mathrm{i}K - \lambda I) = \det(2V - \lambda I) = \det(2a\tilde{L}(0) - \lambda I) = \det(2a\tilde{L}(t) - \lambda I). \tag{39}$$

Finally it is well known, see [3, 21], that

$$\det(2a\tilde{L}(t) - \lambda I) = \sum_{k=0}^{N} (-1)^k (2a)^{N-k} J_{N-k}\lambda^k \tag{40}$$

where integrals of motion J_k's are constructed from matrix $\tilde{L}$ as sums of its principal minors of order K.

References

[1] Calogero F 1969 *J. Math. Phys.* **10** 2191
Calogero F 1969 *J. Math. Phys.* **10** 2197
Calogero F 1971 *J. Math. Phys.* **12** 419
Calogero F 1974 *J. Math. Phys.* **15**1425

[2] Sutherland B 1971 *J. Math. Phys.* **12** 246
Sutherland B 1971 *J. Math. Phys.* **12** 251
Sutherland B 1971 *Phys. Rev.* A **4** 2019
Sutherland B 1972 *Phys. Rev.* A **5** 1372

[3] Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313
Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 313

[4] Moser J 1975 *Adv. Math.* **16** 1

[5] Andric I, Jevicki A and Levine M 1983 *Nucl. Phys.* B **215** 307
Jevicki A 1992 *Nucl. Phys.* B **376** 75

[6] Minahan J A and Polychronakos A P 1993 *Phys. Lett.* B **312** 155

Minahan J A and Polychronakos A P 1994 *Nucl. Phys.* B **336** 288
[7] Polychronakos A P 1989 *Nucl. Phys.* B **324** 597
Jsakov S B and Leinaas J M 1996 *Nucl. Phys.* B **463** 194
Leinaas J M Generalized statistics and the algebra of observables *Preprint* hep-th/9611167
[8] Kawakami N 1993 *Phys. Rev. Lett.* **71** 275
Azuma O and Iso S 1994 *Phys. Lett.* B **331** 107
[9] Haldane F D M 1988 *Phys. Rev. Lett.* **60** 635
Shastry B S 1988 *Phys. Rev. Lett.* **60** 639
[10] Ujino H, Hikami K and Wadati M 1992 *J. Phys. Soc. Japan* **61** 3425
Ujino H, Hikami K and Wadati M 1993 *J. Phys. Soc. Japan* **62** 3035
Ujino H and Wadati M 1995 *J. Phys. Soc. Japan* **64** 4121
Wadati M, Hikami K and Ujino H 1993 *Chaos, Solitons Fractals* **3** 627
[11] Babelon O and Viallet C-M 1990 *Phys. Lett.* B **237** 412
Avan J and Talon M 1993 *Phys. Lett.* B **303** 33
Avan J, Babelon O and Talon M 1994 *Alg. Anal.* **6** 67
Sklyanin E K 1994 *Alg. Anal.* **6** 227
Barden H W and Suzuki T 1994 *Lett. Math. Phys.* **30** 147
[12] Avan J and Jevicki A 1991 *Phys. Lett.* B **272** 17
Avan J and Jevicki A 1992 *Commun. Math. Phys.* **150** 149
Caracciolo R, Lerda A and Zemba G R 1995 *Phys. Lett.* B **352** 304
Sen D and Bhaduri R K 1997 *Ann. Phys.* **260** 203–23
[13] Ujino H and Wadati M 1994 *J. Phys. Soc. Japan* **63** 3585
Ujino H and Wadati M 1993 *J. Phys. Soc. Japan* **62** 4203
Ujino H and Wadati M 1994 *Phys. Rev. Lett.* **73** 1191
Narayana V and Sivakumar M 1996 *Mod. Phys. Lett.* A **11** 763
[14] Wojciechowski S 1983 *Phys. Lett.* **95A** 279
[15] Kuznetsov V 1996 *Phys. Lett.* A **218** 212
[16] Gonera C and Kosiński P A note on super integrability of quantum Calogero model *J. Math. Phys.* to appear
[17] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[18] Landau L D and Lifshitz E M 1971 *Mechanics* (Oxford: Pergamon)
[19] Perelomov A M 1992 *Integrable Systems of Classical Mechanics and Lie Algebras* (Boston, MA: Birkhäuser)
[20] Gonera C 1998 *Phys. Lett.* A **237** 365–8
[21] Calogero P and Ragnisco O 1975 *Lett. Nuovo Cimento* **38**